

TRANSVERSELY AFFINE HOLOMORPHIC FOLIATIONS OF ARBITRARY CODIMENSION - I

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ABSTRACT. We study holomorphic foliations with an affine homogeneous transverse structure. We give a friendly characterization of the case of transversely affine foliations in terms of matrix valued pairs of differential forms. This leads naturally to the study of the case of foliations with singularities. A first extension theorem is then proved in the generic singularities framework.

1. INTRODUCTION

The study of the geometry of foliations often is related to the study of their transverse structure. Among the most comprehensible structures are those given by actions of Lie groups on some homogeneous space. This is the case of the so called *transversely homogeneous foliations* as introduced by Blumenthal ([1, 5]). One of the first cases of such a class of foliations, is the class of transversely affine foliations. Such foliations have been studied in the smooth real codimension one case by Bobo Seke in [7]. In [13] the author considers the case of codimension one holomorphic foliations with singularities. A classification is given for such objects on complex projective spaces.

In this paper we consider the case of arbitrary codimension. We focus on the holomorphic case, already aiming the case of foliations with singularities. Nevertheless, most of the material in the first sections also holds in the (non-singular) smooth case. In few words, our aim is to introduce the first ingredients in the study of the case of transversely homogeneous holomorphic foliations with singularities.

1.1. Transversely affine foliations. Let us clearly state the notions we use. The following definition is found in [1] or in [5] pp. 245. We adapt it to the holomorphic case:

Definition 1.1 (transversely homogeneous foliation). Let \mathcal{F} be a holomorphic foliation on a complex manifold P . Let G be a simply-connected Lie group and $H \subset G$ be a connected closed subgroup of G . We say that \mathcal{F} is *transversely homogeneous* in P of model G/H if P admits an open cover $\bigcup_{i \in I} U_i = P$ with holomorphic submersions $y_i: U_i \rightarrow G/H$ satisfying: (i) $\mathcal{F}|_{U_i}$ is defined by y_i , (ii) In each $U_i \cap U_j \neq \emptyset$ we have $y_i = g_{ij} \circ y_j$ for some locally constant map $g_{ij}: U_i \cap U_j \rightarrow G$.

Notice that the group G acts on the quotient $P = G/H$ by left translations. In particular, we have:

Definition 1.2. A holomorphic codimension- q foliation \mathcal{F} on M^n is *transversely affine* if there is a family $\{Y_i: U_i \rightarrow \mathbb{C}^q\}_{i \in I}$ of holomorphic submersions $Y_i: U_i \rightarrow \mathbb{C}^q$ defined in open sets $U_i \subset M$, defining $\mathcal{F}|_{U_i}$, covering $M = \bigcup_{i \in I} U_i$ and such that for each $U_i \cap U_j \neq \emptyset$ we have $Y_i = A_{ij}Y_j + B_{ij}$ for some locally constant maps $A_{ij}: U_i \cap U_j \rightarrow \text{GL}_q(\mathbb{C})$, $B_{ij}: U_i \cap U_j \rightarrow \mathbb{C}^q$.

1.2. Integrable systems and foliations. Recall that a system of holomorphic 1-forms $\Omega := \{\Omega_1, \dots, \Omega_q\}$ in an open set $U \subset M$ is *integrable* if for every $j \in \{1, \dots, q\}$ we have $d\Omega_j \wedge \Omega_1 \wedge \dots \wedge \Omega_q = 0$ in U . If such a system of forms has maximal rank at each point, then it defines a codimension q holomorphic foliation $\mathcal{F}(\Omega)$ on U . The foliation is given by the integrable distribution of $(n - q)$ -planes $\text{Ker}(\Omega) := \bigcap_{j=1}^q \text{Ker}(\Omega_j)$ where given $p \in M$ we define $\text{Ker}(\Omega_j)(p) := \{v \in T_p(M) : \Omega_j(p) \cdot v = 0\}$. Two such maximal rank integrable systems Ω and Ω' define the same foliation in U if, and only if, we have $\Omega_i = \sum_{j=1}^q a_{ij} \Omega_j$ for some holomorphic functions a_{ij} in U , with the property that the $q \times q$ matrix $A = (a_{ij})_{i,j=1}^q$ is nonsingular at each point of U . Given a system $\{\Omega_1, \dots, \Omega_q\}$ as above, we define a $q \times 1$ matrix valued 1-form Ω as having rows given by $\Omega_1, \dots, \Omega_q$. We denote by $\mathcal{F}(\Omega)$ the foliation defined by this system.

Let us now introduce some notation. Given a $k \times \ell$ and a $\ell \times s$ matrix valued 1-form $A = (a_{ij})$ and $B = (b_{jt})$ respectively, we may define the wedge product $A \wedge B$ in the natural way, as the $k \times s$ matrix valued 1-form $A \wedge B$ whose entry at the position (i, t) is the 2-form $\sum_{j=1}^{\ell} a_{ij} \wedge b_{jt}$. In the same way we may define the exterior derivative dA as the $k \times \ell$ matrix valued 2-form whose entry at the position (i, j) is the 2-form da_{ij} .

Example 1.1. Let $\mathcal{F}_1, \dots, \mathcal{F}_q$ be transversely affine codimension-one foliations on M^n , which are transverse everywhere. Then the intersection foliation $\bigcap_{i=1}^q \mathcal{F}_i$ is a codimension- q foliation on M which is transversely affine. Indeed, assume that \mathcal{F}_j is given by some holomorphic integrable 1-form Ω_j in M . According to [13] Chapter I Proposition 1.1 we have $d\Omega_j = \eta_j \wedge \Omega_j$, $d\eta_j = 0$, for some holomorphic 1-form η_j in M . Define Ω as the $q \times 1$ matrix valued 1-form in M having $\Omega_1, \dots, \Omega_q$ as rows. Also define η the $q \times q$ diagonal matrix valued holomorphic 1-form in M having η_1, \dots, η_q in its diagonal. Then, in the above notation we have $d\Omega = \eta \wedge \Omega$. Since η is diagonal, we have $d\eta = 0 = \eta \wedge \eta$.

As for the general case we have the following description:

Theorem 1.1. *Let \mathcal{F} be a holomorphic codimension- q foliation on M . The foliation \mathcal{F} is transversely affine in M if, and only if, there exist an open cover $\bigcup_{i \in I} U_i = M$ and holomorphic $q \times 1$,*

$q \times q$ matrix valued 1-forms Ω_i, η_i in U_i , $\forall i \in I$, satisfying:

- a) $\mathcal{F}|_{U_i} = \mathcal{F}(\Omega_i)$
- b) $d\Omega_i = \eta_i \wedge \Omega_i$ and $d\eta_i = \eta_i \wedge \eta_i$
- c) if $U_i \cap U_j \neq \emptyset$ then we have $\Omega_i = G_{ij} \cdot \Omega_j$ and $\eta_i = \eta_j + dG_{ij} \cdot G_{ij}^{-1}$ for some holomorphic $G_{ij}: U_i \cap U_j \rightarrow \text{GL}_q(\mathbb{C})$.

Moreover, two such collections $\{(\Omega_i, \eta_i, U_i)\}_{i \in I}$ and $\{(\Omega'_i, \eta'_i, U_i)\}_{i \in I}$ define the same affine transverse structure for \mathcal{F} , if and only if, we have $\Omega'_i = G_i \cdot \Omega_i$ and $\eta'_i = \eta_i + dG_i \cdot G_i^{-1}$ for some holomorphic $G_i: U_i \rightarrow \text{GL}_q(\mathbb{C})$.

Remark 1.1. Theorem 1.1 is stated in a much more abstract context by Blumenthal (see Theorem 2 page 144 as well as its Corollary 3.2 page 149). Nevertheless, it is required some triviality hypothesis on principal fiber-bundles of structural group G/H , over the manifold M (see also [5] Prop. 3.6 pp. 249-250). In our case, we will obtain it from some explicit computations and some classical results on Lie groups (see Theorem 2.1).

In the final section we prove that an extension result for the pair (Ω, η) associate to an affine transverse structure off some codimension one divisor, under the presence of generic singularities for the foliation on the divisor (cf. Theorem 6.1).

2. AUXILIARY RESULTS

We state some results of easy proof which will be used in the proof of Theorem 1.1. We start by the following well-known lemma from real analysis, adapted to the holomorphic case:

Lemma 2.1. *Let $X: U \subset \mathbb{C}^n \rightarrow \mathrm{GL}_q(\mathbb{C})$ be a holomorphic map, then $d(X^{-1}) = -X^{-1} \cdot dX \cdot X^{-1}$.*

Next step is:

Lemma 2.2. *Let $X: U \subset \mathbb{C}^n \rightarrow \mathrm{GL}_q(\mathbb{C})$ be holomorphic and let η be defined diagonal by $\eta = dX \cdot X^{-1}$ then we have $d\eta = \eta \wedge \eta$. Given a holomorphic $q \times q$ matrix valued 1-form η in $U \subset \mathbb{C}^n$, such that $d\eta = \eta \wedge \eta$, and a holomorphic map $G: U \rightarrow \mathrm{GL}_q(\mathbb{C})$, then the 1-form $\tilde{\eta} := \eta + dG \cdot G^{-1}$ satisfies $d\tilde{\eta} = \tilde{\eta} \wedge \tilde{\eta}$.*

Proof. Using Lemma 2.1 we have $d(X^{-1}) = -X^{-1} \cdot dX \cdot X^{-1}$. Thus

$$\begin{aligned} d\eta &= d(dX \cdot X^{-1}) = d(dX) \wedge X^{-1} + (-1)dX \wedge d(X^{-1}) \\ &= (-1)dX \wedge (-X^{-1} \cdot dX \cdot X^{-1}) \\ &= (dX \cdot X^{-1}) \wedge (dX \cdot X^{-1}) = \eta \wedge \eta. \end{aligned}$$

As for the second part, we have $d\tilde{\eta} = d\eta + d(dG \cdot G^{-1}) = \eta \wedge \eta + dG \cdot G^{-1} \wedge dG \cdot G^{-1}$. On the other hand $\tilde{\eta} \wedge \tilde{\eta} = (\eta + dG \cdot G^{-1}) \wedge (\eta + dG \cdot G^{-1}) = \eta \wedge \eta + \eta \wedge dG \cdot G^{-1} + dG \cdot G^{-1} \wedge \eta + dG \cdot G^{-1} \wedge dG \cdot G^{-1} = \eta \wedge \eta + dG \cdot G^{-1} \wedge dG \cdot G^{-1}$. \square

Finally, we have:

Lemma 2.3. *Let $G, G': U \subset \mathbb{C}^n \rightarrow \mathrm{GL}_q(\mathbb{C})$ be holomorphic maps. Then we have $dG \cdot G^{-1} = dG' \cdot G'^{-1}$ if and only if $G' = G \cdot A$ for some locally constant $A: U \rightarrow \mathrm{GL}_q(\mathbb{C})$.*

Proof. First we assume that $G' = G \cdot A$ with A locally constant. Thus we have $G^{-1} \cdot G' = A$ and therefore $d(G^{-1} \cdot G') = dA = 0$ in U . This implies $d(G^{-1}) \cdot G' + G^{-1} \cdot d(G') = 0$. Using that $d(G^{-1}) = -G^{-1} \cdot dG \cdot G^{-1}$ we have

$$-G^{-1} \cdot dG \cdot G^{-1} \cdot G' + G^{-1} \cdot dG' = 0.$$

Multiplying on the left this equality by G we obtain

$$-dG \cdot G^{-1} \cdot G' + dG' = 0.$$

Multiplying on the right this last equality by G'^{-1} we obtain

$$-dG \cdot G^{-1} + dG' \cdot (G')^{-1} = 0,$$

which proves the first part. Now we assume that $dG \cdot G^{-1} = dG' \cdot (G')^{-1}$ in U . Define $A = G^{-1} \cdot G'$ so that $G' = G \cdot A$. We only have to show that $dA = 0$ in U .

In fact, we have

$$d(A) = d(G^{-1} \cdot G') = d(G^{-1}) \cdot G' + G^{-1} \cdot d(G').$$

Since $d(G^{-1}) = -G^{-1} \cdot dG \cdot G^{-1}$ we get

$$\begin{aligned} dA &= -G^{-1} \cdot dG \cdot G^{-1} \cdot G' + G^{-1} \cdot dG' \\ &= -G^{-1} \cdot (dG \cdot G^{-1} - dG' \cdot G'^{-1})G'. \end{aligned}$$

Using the hypothesis $dG \cdot G^{-1} = DG' \cdot G'^{-1}$ we obtain $dA = 0$. \square

Let G be a Lie group and $\{\omega_1, \dots, \omega_\ell\}$ be a basis of the Lie algebra of G . Then we have $d\omega_k = \sum_{i < j} c_{ij}^k \omega_i \wedge \omega_j$ for a family constants $\{c_{ij}^k\}$ called the *structure constants* of the Lie algebra in the given basis ([5]). With this we have the classical theorem due to Darboux and Lie below. In few words, it says that maximal rank systems of 1-forms satisfying the same equations are locally pull-back of the group Lie algebra. The map is unique up to left translations in the Lie group.

Theorem 2.1 (Darboux-Lie, [5]). *Let G be a (complex) Lie group of dimension ℓ . Let $\{\omega_1, \dots, \omega_\ell\}$ be a basis of the Lie algebra of G with structure constants $\{c_{ij}^k\}$. Given a maximal rank system of (holomorphic) 1-forms $\Omega_1, \dots, \Omega_\ell$ in a (complex) manifold V , such that $d\Omega_k = \sum_{i,j} c_{ij}^k \Omega_i \wedge \Omega_j$, then:*

- (1) *For each point $p \in V$ there is a neighborhood $p \in U_p \subseteq V$ equipped with a (holomorphic) submersion $f_p: U_p \rightarrow G$ which defines \mathcal{F} in U_p such that $f_p^*(\omega_j) = \Omega_j$ in U_p , for all $j \in \{1, \dots, q\}$.*
- (2) *If V is simply-connected we can take $U_p = V$.*
- (3) *If $U_p \cap U_q \neq \emptyset$ then in the intersection we have $f_q = L_{g_{pq}}(f_p)$ for some locally constant left translation $L_{g_{pq}}$ in G .*

3. TRANSVERSELY AFFINE FOLIATIONS AND DIFFERENTIAL FORMS

The first step in the proof of Theorem 1.1 is:

Proposition 3.1. *Let \mathcal{F} be a holomorphic codimension- q foliation on M . Suppose that \mathcal{F} is defined by some integrable system $\{\Omega_1, \dots, \Omega_q\}$ of holomorphic 1-forms. If \mathcal{F} is transversely affine then there is a $q \times q$ matrix valued holomorphic 1-form $\eta = (\eta_{ij})$ satisfying:*

$$d\Omega = \eta \wedge \Omega, \quad d\eta = \eta \wedge \eta \quad \text{where} \quad \Omega = \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_q \end{pmatrix}$$

Proof. Let $\{\Omega_1, \dots, \Omega_q\}$ be an integrable holomorphic system which defines \mathcal{F} in M and suppose $\{Y_i: U_i \rightarrow \mathbb{C}^q\}_{i \in I}$ is a transversal affine structure for \mathcal{F} in M with

$$Y_i = A_{ij}Y_j + B_{ij} \quad \text{in} \quad U_i \cap U_j \neq \emptyset \quad (1)$$

as in Definition 1.2.

Since the submersions Y_i define \mathcal{F} we can write

$$\Omega = G_i \cdot dY_i \quad (2)$$

in each U_i , for some holomorphic $G_i: U_i \rightarrow \text{GL}_q(\mathbb{C})$. Here $\Omega = \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_q \end{pmatrix}$.

In each $U_i \cap U_j \neq \emptyset$ we have:

$$G_i dY_i = G_j dY_j \quad (3)$$

and as it follows from (1)

$$G_j = A_{ij} \cdot G_i \quad (4)$$

According to Lemma 2.3 this last equality implies:

$$dG_j \cdot G_j^{-1} = dG_i \cdot G_i^{-1} \quad (5)$$

in each $U_i \cap U_j \neq \emptyset$.

This allows us to define η in M by

$$\eta|_{U_i} = dG_i \cdot G_i^{-1}. \quad (6)$$

According to Lemma 2.2 we have $d\eta = \eta \wedge \eta$. We also have in each U_i

$$\begin{aligned} d\Omega &= d(G_i dY_i) = dG_i \wedge dY_i \\ &= dG_i \cdot G_i^{-1} \wedge dY_i \\ &= dG_i \cdot G_i^{-1} \wedge G_i dY_i \\ &= \eta \wedge \Omega. \end{aligned}$$

The pair (Ω, η) satisfies the conditions of the statement. □

Now we study the converse of the proposition above.

Proposition 3.2. *Let \mathcal{F} be a holomorphic condimension- q foliation on M . The foliation \mathcal{F} is transversely affine in M if, and only if, there exist an open cover $\bigcup_{i \in I} U_i = M$ and holomorphic*

$q \times 1$, $q \times q$ matrix valued 1-forms Ω_i, η_i in U_i , $\forall i \in I$, satisfying:

a) $\mathcal{F}|_{U_i} = \mathcal{F}(\Omega_i)$

b) $d\Omega_i = \eta_i \wedge \Omega_i$ and $d\eta_i = \eta_i \wedge \eta_i$

c) if $U_i \cap U_j \neq \emptyset$ then we have $\Omega_i = G_{ij} \cdot \Omega_j$ and $\eta_i = \eta_j + dG_{ij} \cdot G_{ij}^{-1}$ for some holomorphic $G_{ij}: U_i \cap U_j \rightarrow \text{GL}_q(\mathbb{C})$.

Moreover, two such collections $\{(\Omega_i, \eta_i, U_i)\}_{i \in I}$ and $\{(\Omega'_i, \eta'_i, U_i)\}_{i \in I}$ define the same affine transverse structure for \mathcal{F} , if and only if, we have $\Omega'_i = G_i \cdot \Omega_i$ and $\eta'_i = \eta_i + dG_i \cdot G_i^{-1}$ for some holomorphic $G_i: U_i \rightarrow \text{GL}_q(\mathbb{C})$.

In order to prove in details the proposition above we explicitly calculate the Lie algebra of $\text{Aff}(\mathbb{C}^q)$. We consider $\text{GL}_q(\mathbb{C})$ as an open subset of the vector space $M(q \times q, \mathbb{C})$ of complex $q \times q$ matrices. Using this we have:

Lemma 3.1. *The Lie algebra $\text{aff}(\mathbb{C}^q)$ of $\text{Aff}(\mathbb{C}^q)$ has a basis given by $\Omega = X \cdot dY$, $\eta = dX \cdot X^{-1}$ where $X \in \text{GL}_q(\mathbb{C})$ and $Y \in \mathbb{C}^q$ are global coordinates. Furthermore we have $d\Omega = \eta \wedge \Omega$, $d\eta = \eta \wedge \eta$.*

Proof. We denote by $M(q \times q, \mathbb{C})$ the linear space of $q \times q$ complex matrices. Since $\text{GL}_q(\mathbb{C}) \subset M(q \times q, \mathbb{C}) \cong \mathbb{C}^{q^2}$ as an open set, we have a natural global coordinate X in $\text{GL}_q(\mathbb{C})$. Let us denote by Y the natural global coordinate in \mathbb{C}^q . Fixed any element $(X_o, Y_o) \in \text{Aff}(\mathbb{C}^q)$ it defines a left translation by

$$\begin{aligned} L_{(X_o, Y_o)}: \text{GL}_q(\mathbb{C}) \times \mathbb{C}^q &\longrightarrow \text{GL}_q(\mathbb{C}) \times \mathbb{C}^q \\ L_{(X_o, Y_o)}(X, Y) &= (X_o X, X_o Y + Y_o). \end{aligned}$$

Therefore given any vector $(V, W) \in T_{(X_o, Y_o)}(\text{GL}_q(\mathbb{C}) \times \mathbb{C}^q)$ we have $\text{DL}_{(X_o, Y_o)}(X, Y) \cdot (V, W) = (X_o V, X_o W)$. Therefore a basis of the left-invariant vector fields in $\text{Aff}(\mathbb{C}^q)$ is given by:

$$\mathcal{X} = (X, X) = X \cdot \frac{\partial}{\partial X} + X \cdot \frac{\partial}{\partial Y} \in T(\text{Aff}(\mathbb{C}^q)) = \text{GL}_q(\mathbb{C}) \times \mathbb{C}^q.$$

Thus a basis of $\text{aff}(\mathbb{C}^q)$ is given by the dual basis $\{\Omega, \eta\}$ of $\{\mathcal{X}\}$. This shows that

$$\begin{cases} \Omega &= X \cdot dY \\ \eta &= dX \cdot X^{-1} \end{cases}$$

is a basis for $\text{aff}(\mathbb{C}^q)$.

It is now a straightforward calculation to show that $d\Omega = \eta \wedge \Omega$ and $d\eta = \eta \wedge \eta$. \square

Using these two lemmas and Darboux-Lie Theorem (Theorem 2.1) or alternatively, the book of Spivak ([14] Chapter 10, Theorem 17 page 397, Theorem 18 page 398 and Corollary 19 page 400) we obtain:

Corollary 3.1. (a) *Let η be a holomorphic $q \times q$ matrix valued 1-form in M satisfying $d\eta = \eta \wedge \eta$. Then locally in M we have $\eta = dX \cdot X^{-1}$ for some holomorphic $X: U \subset M \rightarrow \text{GL}_q(\mathbb{C})$. If M is simply-connected we can choose $U = M$. Moreover given two such trivializations (X, U) and (\tilde{X}, \tilde{U}) with $U \cap \tilde{U} \neq \emptyset$ connected then we have $\tilde{X} = X \cdot A$ for some $A \in \text{GL}_q(\mathbb{C})$.*

b) *Let Ω, η be holomorphic $q \times 1, q \times q$ matrix valued 1-forms in M satisfying $d\Omega = \eta \wedge \Omega$ and $d\eta = \eta \wedge \eta$. Then given any point $p \in M$ and given any simply-connected open neighborhood $p \in U_p \subset M$ we have $\Omega = X \cdot dY, \eta = dX \cdot X^{-1}$ for some holomorphic $\pi_p = (X, Y): U_p \rightarrow \text{GL}_q(\mathbb{C}) \times \mathbb{C}^q$. Furthermore in each connected component of $U_p \cap \tilde{U}_{\tilde{p}} \neq \emptyset$ we have $\pi_q = L \circ \pi_{\tilde{p}}$ for some left-translation $L: \text{GL}_q(\mathbb{C}) \times \mathbb{C}^q \rightarrow \text{GL}_q(\mathbb{C}) \times \mathbb{C}^q$. In particular if M is simply-connected we can choose $U_p = M$.*

The proof of Proposition 3.2 is now an easy consequence of Corollary 3.1 above and of the arguments used in the proof of Proposition 3.1.

Proof of Proposition 3.2. Proposition 3.1 shows that if \mathcal{F} is transversely affine in M then we can construct collections (Ω_j, η_j) in open subsets $U_j \subset M$ covering M as stated. Conversely assume that (Ω, η) is a pair, where Ω defines \mathcal{F} in M , like in the statement. Since η is holomorphic and satisfies $d\eta = \eta \wedge \eta$ in M , there exists an open cover $\bigcup U_i$ of M there are holomorphic $G_i: U_i \rightarrow \text{GL}_q(\mathbb{C})$ such that $\eta|_{U_i} = dG_i \cdot G_i^{-1}$ (Corollary 3.1 (a)).

Now, from condition $d\Omega = \eta \wedge \Omega$ we have

$$\begin{aligned} d(G_i^{-1} \cdot \Omega) &= -G_i^{-1} dG_i G_i^{-1} \wedge \Omega + G_i^{-1} d\Omega \\ &= -G_i^{-1} \eta \wedge \Omega + G_i^{-1} \eta \wedge \Omega = 0 \end{aligned}$$

and therefore $G_i^{-1} \cdot \Omega = dY_i$ for some holomorphic $Y_i: U_i \rightarrow \mathbb{C}^q$ which is a submersion.

Therefore we have $\Omega = G_i dY_i$ in U_i . Moreover according to Lemma 2.3 we have $G_i^{-1} G_j = A_{ij}$ for some locally constant $A_{ij}: U_i \cap U_j \rightarrow \text{GL}_q(\mathbb{C})$, in each $U_i \cap U_j \neq \emptyset$.

Therefore $G_i dY_i = \Omega = G_j dY_j = G_i A_{ij} dY_j$ so that $dY_i = A_{ij} dY_j = d(A_{ij} Y_j)$ in each $U_i \cap U_j \neq \emptyset$ and thus $Y_i = A_{ij} Y_j + B_{ij}$ for some locally constant $B_{ij}: U_i \cap U_j \rightarrow \mathbb{C}^q$. This shows that \mathcal{F} is transversely affine in M . \square

Theorem 1.1 is now a straightforward consequence of Propositions 3.1 and 3.2.

4. A SUSPENSION EXAMPLE

The following example generalizes Example 1.5 of Chapter I in [13].

Example 4.1. We will define a transversely affine codimension- q holomorphic foliation on a compact manifold by the suspension method:

Let M be a complex manifold and let w be a $q \times 1$ holomorphic matrix valued 1-form on M , closed and satisfying $f^*w = Aw$ for some biholomorphism $f: M \rightarrow M$ and some hyperbolic matrix $A \in \mathrm{GL}_q(\mathbb{C})$. Define Ω and η in the product $M \times \mathrm{GL}_q(\mathbb{C})$ by $\Omega(x, T) = T.w(x)$ and $\eta(x, T) = dT.T^{-1}$.

Then we have

$$\begin{aligned} d\Omega(x, T) &= dT \wedge w(x) + T dw(x) = \\ &= dT \wedge w(x) = dT.T^{-1} \wedge Tw(x) = \\ &= \eta(x, T) \wedge \Omega(x, T) \end{aligned}$$

and also,

$$\begin{aligned} d\eta(x, T) &= d(dT.T^{-1}) = dT.T^{-1} \wedge dT.T^{-1} = \\ &= \eta(x, T) \wedge \eta(x, T). \end{aligned}$$

Moreover the biholomorphism $F: M \times \mathrm{GL}_q(\mathbb{C}) \rightarrow M \times \mathrm{GL}_q(\mathbb{C})$ defined by $F(x, T) = (f(x), T.A^{-1})$ satisfies

$$F^*\Omega = TA^{-1}f^*w = TA^{-1}Aw = Tw = \Omega$$

and

$$F^*\eta = d(TA^{-1}) \cdot (TA^{-1})^{-1} = dT.T^{-1} = \eta.$$

Thus, by Theorem 1.1 the pair Ω, η induces a codimension- q non-singular holomorphic foliation $\tilde{\mathcal{F}}$ which is transversely affine in $M \times \mathrm{GL}_q(\mathbb{C})$. This foliation induces a codimension- q non-singular foliation \mathcal{F} on the quotient manifold $V = (M \times \mathrm{GL}_q(\mathbb{C}))/\mathbb{Z}$ by the action $\mathbb{Z} \times (M \times \mathrm{GL}_q(\mathbb{C})) \rightarrow M \times \mathrm{GL}_q(\mathbb{C})$, $\eta, (x, T) \mapsto (f^n(x), T.A^{-n})$. This last foliation \mathcal{F} inherits an affine transverse structure from $\tilde{\mathcal{F}}$.

5. HOLOMORPHIC FOLIATIONS WITH SINGULARITIES

A codimension q holomorphic foliation with singularities \mathcal{F} on a complex manifold M of dimension $n \geq 2$ is defined as a pair $(\mathcal{F}_0, \mathrm{sing}(\mathcal{F}))$, where $\mathrm{sing}(\mathcal{F}) \subset M$ is an analytic subset of codimension $\geq q + 1$, and a holomorphic foliation \mathcal{F}_0 in the classical, in the open manifold $M \setminus \mathrm{sing}(\mathcal{F})$. Then, all the notions for \mathcal{F} are defined in terms of \mathcal{F}_0 . For instance, the leaves of \mathcal{F} are defined as the leaves of \mathcal{F}_0 , and their holonomy groups are defined in the same way. We may assume that the *singular set* $\mathrm{sing}(\mathcal{F})$ is saturated in the sense that there is no other pair $\mathcal{F}' = (\mathcal{F}'_0, \mathrm{sing}(\mathcal{F}'))$ with $\mathrm{sing}(\mathcal{F}') \subsetneq \mathrm{sing}(\mathcal{F})$ and such that \mathcal{F}'_0 coincides with \mathcal{F}_0 on $M \setminus \mathrm{sing}(\mathcal{F})$.

Definition 5.1. A codimension- q holomorphic foliation with singularities \mathcal{F} on M^n is said to be *transversely affine* if there is a family $\{Y_i: U_i \rightarrow \mathbb{C}^q\}_{i \in I}$ of holomorphic submersions $Y_i: U_i \rightarrow \mathbb{C}^q$ defined in open sets $U_i \subset M$, defining \mathcal{F} , and satisfying $M \setminus \mathrm{sing}(\mathcal{F}) = \bigcup_{i \in I} U_i$ and with affine relations $Y_i = A_{ij}Y_j + B_{ij}$ for some $A_{ij}: U_i \cap U_j \rightarrow \mathrm{GL}_q(\mathbb{C})$, $B_{ij}: U_i \cap U_j \rightarrow \mathbb{C}^q$ locally constant in each $U_i \cap U_j \neq \emptyset$.

We usually distinguish two cases in the definition above: the codimension one and the dimension one cases. Since we are interested in the codimension ≥ 2 case, we shall focus on the second case.

5.1. Generic singularities. In this paragraph we introduce what we will consider as generic type of a singularity for a codimension- $q \geq 2$ foliation. Given a holomorphic foliation with singularities \mathcal{F} on a complex manifold M , the singular set of \mathcal{F} is an analytic subset $\text{sing}(\mathcal{F}) \subset M$ of codimension ≥ 2 , also having dimension $\dim \text{sing}(\mathcal{F}) \leq \dim(\mathcal{F})$. In particular, it can have a component of dimension $\dim(\mathcal{F})$, as well as a component of dimension $\dim(\mathcal{F}) - 1$. As for this second case, by intersecting with appropriate transverse small discs we may consider the following model of generic singularity:

5.1.1. Isolated singularities.

Definition 5.2. Let \mathcal{F} be a germ of an *isolated* one-dimensional foliation singularity at the origin $0 \in \mathbb{C}^{q+1}$. The singularity is called *Poincaré non-resonant* if the convex hull of the set of eigenvalues of the linear part $DX(0)$ does not contain the origin, and there is no resonant $\lambda_j = n_1\lambda_1 + \dots + n_{q+1}\lambda_{q+1}$ for $n_1, \dots, n_{q+1} \in \mathbb{N}$. In this case, by Poincaré linearization theorem ([2], [4]) the singularity *linearizable without resonances* ([11]): it is given in some neighborhood U of $0 \in \mathbb{C}^{q+1}$ by

a holomorphic vector field X which is analytically linearizable as $X = \sum_{j=1}^{q+1} \lambda_j z_j \frac{\partial}{\partial z_j}$, with eigenvalues

$\lambda_1, \dots, \lambda_{q+1}$ satisfying the following non-resonance hypothesis:

If $n_1, \dots, n_{q+1} \in \mathbb{Z}$ are such that $\sum_{j=1}^{q+1} n_j \lambda_j = 0$, then $n_1 = n_2 = \dots = n_{q+1} = 0$.

In the above situation, define 1-forms $\omega^1, \dots, \omega^q$ on $U \setminus \Lambda$ by setting $\omega^\nu(X) = 0$ and $\omega^\nu = \sum_{j=1}^{q+1} \alpha_j^\nu \frac{dz_j}{z_j}$, where $\nu = 1, \dots, q$ and $\alpha_j^\nu \in \mathbb{C}$. From this we get the following system of equation $\sum_{j=1}^{q+1} \alpha_j^\nu \lambda_j = 0$, $\nu = 1, \dots, q$. the equation $\sum_{j=1}^{q+1} \lambda_j z_j = 0$ defines a hyperplane in \mathbb{C}^{q+1} implies that we can choose q linearly independent vectors $\vec{\alpha}_1, \dots, \vec{\alpha}_q$ say $\vec{\alpha}_\nu = (\alpha_1^\nu, \dots, \alpha_q^\nu, \alpha_{q+1}^\nu) \in \mathbb{C}^{q+1}$ so that $\sum_{j=1}^{q+1} \alpha_j^\nu \lambda_j = 0$, $\nu = 1, \dots, q$. and therefore the system $\omega^1, \dots, \omega^q$ has maximal rank q outside the coordinate hyperplanes.

Lemma 5.1 ([11]). *Let $f(z)$ be a holomorphic function on the set $U \setminus \{z_1 \dots z_{q+1} = 0\}$, where U is a connected neighborhood of the origin in \mathbb{C}^{q+1} . Then $f(z)$ is constant provided that $df \wedge \omega^1 \wedge \dots \wedge \omega^q = 0$.*

Definition 5.3 (type II generic singularities). A singularity $p \in \text{sing}(\mathcal{F})$ will be called *type II generic singularity* if p belongs to a smooth part of the set $\text{sing}(\mathcal{F})$, where:

- There is a unique branch $\text{sing}(\mathcal{F})_p \subset \text{sing}(\mathcal{F})$ through p .
- $\dim \text{sing}(\mathcal{F})_p = \dim(\mathcal{F}) - 1$
- For some (and therefore for every) transverse disc Σ_p , with $\Sigma_p \cap \text{sing}(\mathcal{F})_p = \Sigma_p \cap \text{sing}(\mathcal{F}) = \{p\}$, of dimension $q+1$, the induced foliation $\mathcal{F}|_{\Sigma_p}$ exhibits an isolated non-resonant Poincaré type singularity at the origin p .

5.1.2. Non-isolated singularities. Now we focus on the components of the singular set that cannot be reduced to isolated singularities by transverse sections. Let us first recall that some notions for codimension one foliations. Given a codimension-one holomorphic foliation with singularities \mathcal{F} on a complex manifold M , a singular point $p \in \text{sing}(\mathcal{F})$ is a *Kupka-type singularity* (cf. [6, 13]), if \mathcal{F} is given in some neighborhood U of p by a holomorphic integrable 1-form ω , such that $\omega(p) = 0$, $d\omega(p) \neq 0$. In this case, if U is small enough, there exists a system of local coordinates $(x, y, z_1, \dots, z_{n-2}) \in U$ of M , centered at p , such that $\mathcal{F}|_U$ is given by $\alpha(x, y) = 0$, for some holomorphic 1-form $\alpha = A(x, y)dx + B(x, y)dy$. The 1-form α , so called the *transverse type of \mathcal{F} at*

p , has an isolated singularity at the origin $0 \in \mathbb{C}^2$ and satisfies $d\alpha(0) \neq 0$. The generic type is then defined as follows: We shall say that a singularity $p \in \text{sing}(\mathcal{F})$ is *Poincaré type* if it is Kupka type and its corresponding transverse type is of the form $xdy - \lambda ydx + hot = 0$, $\lambda \in \mathbb{C} \setminus (\mathbb{R}_- \cup \mathbb{Q}_+)$. The reasons for this are based on the classification of singularities of germs of foliations in dimension two (see [12], [3]). In this case, the singularity $\alpha(x, y) = 0$ is analytically linearizable, so that there are coordinates $(x, y, z_1, \dots, z_{n-2})$ as above, such that \mathcal{F} is given in these coordinates by $xdt - \lambda ydx = 0$. Let us now motivate our second type of generic singularity for codimension $q \geq 2$ foliations, by discussing an example:

Example 5.1. Let $\mathcal{F}_1, \dots, \mathcal{F}_q$ be holomorphic singular codimension one foliations on a complex manifold M of dimension $q + 1$. Assume that the foliations \mathcal{F}_j are transverse outside the union of their singular sets and their set of tangent points. Then we can define in the natural way the *intersection foliation* $\mathcal{F} = \bigcap_{j=1}^q \mathcal{F}_j$ (as in Example 1.1) whose leaves are obtained as the connected components of the intersection of the leaves of $\mathcal{F}_1, \dots, \mathcal{F}_q$ through points of M and has singular set $\text{sing}(\mathcal{F}) = \bigcup_{j=1}^q \text{sing}(\mathcal{F}_j) \cup T_2$ where T_2 is the union of the codimension ≥ 2 components of the set of tangent points of the foliations. Suppose that \mathcal{F}_j has only Poincaré type singularities, as defined above. Then, given any point $p \in \text{sing}(\mathcal{F}_j) \setminus \bigcup_{i \neq j} \text{sing}(\mathcal{F}_i)$, there exists a local chart $(x, y, z_1, \dots, z_{n-2}) \in U$ of M , centered at p , such that $\mathcal{F}_j|_U$ is given by

$$xdy - \lambda ydx = 0, \quad \lambda \in \mathbb{C} \setminus (\mathbb{R}_- \cup \mathbb{Q}_+)$$

and for each $i \neq j$, $\mathcal{F}_i|_U$ is regular given by $dz_{k_i} = 0$ for some $k_i \in \{1, \dots, n-2\}$.

Definition 5.4 (type I generic singularities). Let \mathcal{F} be a codimension- q foliation on M^n . A singularity $p \in \text{sing}(\mathcal{F})$ is a *type I generic singularity*, if p belongs to a smooth part of the set $\text{sing}(\mathcal{F})$, where:

- There is a unique branch $\text{sing}(\mathcal{F})_p \subset \text{sing}(\mathcal{F})$ through p .
- $\dim \text{sing}(\mathcal{F})_p = \dim(\mathcal{F})$
- There is a local chart $(x, y, z_1, \dots, z_{n-2}) \in U$ of M , centered at p , such that $\mathcal{F}|_U$ is given by

$$xdy - \lambda ydx = 0, \quad \lambda \in \mathbb{C} \setminus (\mathbb{R}_- \cup \mathbb{Q}_+)$$

and $dz_j = 0$, $j = 1, \dots, q-1$.

Therefore in a neighborhood of p , the foliation \mathcal{F} has the structure of the *intersection* (not product) of a singular linear foliation $xdy - \lambda ydx = 0$ on $(\mathbb{C}^2, 0)$ and $q-1$ regular trivial foliations.

We have $s(\mathcal{F}) \cap U = \{(x, y, z_1, \dots, z_{n-2}) \in U \mid x = y = 0\}$. If we define $\Lambda = \{xy = 0\} \cap \{z_1 = \dots = z_{q-1} = 0\}$ then Λ consists of two codimension- q invariant local submanifolds $\Lambda_1 \cup \Lambda_2$ which intersect transversely at the point $p = \Lambda_1 \cap \Lambda_2$.

6. EXTENDING AFFINE TRANSVERSE STRUCTURES WITH POLES

Now consider the following situation:

- (1) \mathcal{F} is a codimension- q singular foliation on M ,
- (2) $\Lambda \subset M$ is an analytic irreducible invariant subvariety of codimension- q (i.e., $\Lambda \setminus \text{sing}(\mathcal{F})$ is a leaf of \mathcal{F}),

- (3) There are analytic codimension-one subvarieties $S_1, \dots, S_q \subset M$ such that Λ is an irreducible component of $\bigcap_{j=1}^q S_j$ and S_j is foliated by \mathcal{F} , $j = 1, \dots, q$.

Under these assumptions we make the following definition:

Definition 6.1. Let $\{\Omega_1, \dots, \Omega_q\}$ be an integrable system of holomorphic 1-forms defining \mathcal{F} . A $q \times q$ matrix valued meromorphic 1-form η defined in a neighborhood of Λ is said to be a *partially-closed logarithmic derivative adapted to Ω along Λ* if:

- $d\Omega = \eta \wedge \Omega$ and η is partially-closed, $d\eta = \eta \wedge \eta$, meromorphic with simple poles,
- $(\eta)_\infty = \bigcup_{j=1}^q S_j$, a union of irreducible codimension one analytic subsets $S_j \subset V$ in a neighborhood V of Λ ,
- given any regular point $p \in \Lambda \setminus \text{sing}(\mathcal{F})$ there exists a local chart $(y_1, \dots, y_q, z_1, \dots, z_{n-q}) \in U$ for M , centered at p , such that:

$$\begin{aligned} U \cap S_j &= \{y_j = 0\}, \quad j = 1, \dots, q \\ \Omega &= G.dY \quad \text{and} \\ \eta &= dG.G^{-1} + \sum_{j=1}^q A_j \cdot \frac{dy_j}{y_j} \quad \text{where} \\ Y &= \begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix}, \end{aligned}$$

$G: U \rightarrow \text{GL}_q(\mathbb{C})$ is holomorphic and A_j is a constant $q \times q$ complex matrix.

The matrix A_j is called the *residue matrix* of η with respect to S_j .

In what follows we consider the problem of extending a form η from an affine transverse structure of \mathcal{F} , an analytic invariant hypersurface. The existence of such extension, as adapted closed logarithmic derivatives, is then assured by the following result:

Theorem 6.1 (Extension Lemma). *Let \mathcal{F} , Λ be as above. Suppose:*

- (1) $\text{sing}(\mathcal{F}) \cap \Lambda$ is nonempty and consists of type I and type II generic singularities, and singularities where $\dim \text{sing}(\mathcal{F}) \leq \dim(\mathcal{F}) - 2$.
- (2) There exists a differential 1-form η defined in some neighborhood V of Λ minus Λ and its local separatrices which defines a transverse affine structure for \mathcal{F} in this set $V \setminus (\Lambda \cup \text{sep}(\Lambda))$, in the sense of Proposition 3.1.

Then η extends meromorphically to a neighborhood of Λ as an adapted form (in the sense of Definition 6.1) to Ω along Λ .

We will extend η to Λ through the singularities of \mathcal{F} in Λ . According to classical Hartogs' extension theorem ([8, 9]), this implies the extension to Λ . Choose $p \in \text{sing}(\mathcal{F}) \cap \Lambda$ and choose local coordinates $(x, y, z_1, \dots, z_{n-2}) \in U$, centered at p , as in Definition 5.4.

Lemma 6.1. *Let \mathcal{F} be a codimension q holomorphic foliation with singularities, defined in an open polydisc $U \subset \mathbb{C}^{q+n}$, with a type I generic singularity or a type II generic singularity at the origin $0 \in \text{sing}(\mathcal{F}) \subset U$. Assume that \mathcal{F} is transversely affine in $U \setminus \Lambda$, where $\Lambda \subset U$ is a finite union of*

irreducible invariant hypersurfaces, each one containing the origin. Assume that \mathcal{F} is given in U by a holomorphic $q \times 1$ matrix 1-form Ω in U with a $q \times q$ matrix 1-form η in U satisfying:

$$d\Omega = \eta \wedge \Omega, \quad d\eta = \eta \wedge \eta.$$

Then η extends meromorphically to a neighborhood of Λ as a partially-closed logarithmic derivative adapted to Ω along Λ (in the sense of Definition 6.1).

Proof. For the sake of simplicity of the notation we will assume that \mathcal{F} has codimension $q = 2$ and the ambient has dimension $q + 1 = 3$. Let us also assume that the singularity is isolated, i.e., of non-resonant Poincaré of type II. The general case is pretty similar. Let then $X = \sum_{j=1}^3 \lambda_j x_j (\partial/\partial x_j)$ be a holomorphic vector field defining \mathcal{F} in suitable coordinates $(x_1, x_2, x_3) \in U'$, in a connected neighborhood $0 \in U' \subset U$ of $0 \in \mathbb{C}^3$, with $\{\lambda_1, \lambda_2, \lambda_3\}$ linearly independent over \mathbb{Q} . Given complex numbers a_1, a_2, a_3 we define a closed 1-form $\omega = \sum_{k=1}^3 a_k dx_k/x_k$. Then $\omega(X) = 0$ if and only if $\sum_{k=1}^3 a_k \lambda_k = 0$. Thus, we can choose 1-forms ω_1, ω_2 given by $\omega_j = \sum_{k=1}^3 a_k^j dx_k/x_k$, $a_k^j \in \mathbb{C}$, such that: ω_1 and ω_2 are linearly independent in the complement of $\cup_{j=1}^3 (x_j = 0)$ and $\Theta_j(X) = 0, j = 1, 2$.

Once we fix such 1-forms, the foliation \mathcal{F} is defined by the integrable system of meromorphic 1-forms $\{\omega_1, \omega_2\}$ in U . Notice that the polar set of the ω_j in U' consists of the coordinate hyperplanes $\{x_i = 0\} \subset U', i = 1, 2, 3$. Let Ω_0 be the 2×1 meromorphic matrix valued 1-form given by the system $\{\omega_1, \omega_2\}$.

Claim 6.1. *Let η_0 be a 2×2 holomorphic matrix valued 1-form defined in $U' \setminus \bigcup_{i=1}^3 \{x_i = 0\}$, such that $d\Omega_0 = \eta_0 \wedge \Omega_0$, $d\eta_0 = \eta_0 \wedge \eta_0$. Then:*

- (1) η_0 is closed, $d\eta_0 = 0$.
- (2) The matrix valued 1-form η_0 extends to a meromorphic matrix valued 1-form in U' , having polar divisor of order one in U' .
- (3) The extension of η_0 is adapted to Ω_0 along Λ .

Let us see how the claim proves the lemma. Indeed, as for the original forms Ω and η we have $\Omega = G\Omega_0$ for some holomorphic matrix $G: \tilde{U} \rightarrow \text{GL}_q(\mathbb{C})$. Thus if we define $\eta_0 := \eta - dG \cdot G^{-1}$ then we are in the situation of the above claim. Thus we conclude that η extends to U' as a closed meromorphic 1-form with simple poles and polar divisor consisting of the coordinate planes. Therefore, the same conclusion of the above claim holds for η and we prove the lemma.

Proof of the claim. Since each ω_j is closed the matrix form Ω_0 is closed. From $d\Omega_0 = \eta_0 \wedge \Omega_0$ we have $\eta_0 \wedge \Omega_0 = 0$. Now we observe that there are holomorphic 2×2 scalar matrices M_1, M_2 defined in $U' \setminus \{x_1 x_2 x_3 = 0\}$, such that $\eta_0 = M_1 \omega_1 + M_2 \omega_2$, where the multiplication of the matrix by the 1-form is the standard scalar type multiplication. Indeed, it is enough to complete the pair ω_1, ω_2 into a basis of the space of holomorphic 1-forms and express η_0 in this basis. Then the condition $\eta_0 \wedge \Omega_0$ means that the coefficients of η_0 in the other elements of the basis are all identically zero.

For any holomorphic 2×2 scalar (holomorphic) matrix M and a 2×1 matrix valued 1-form Ω we have the easily verified formula for the exterior derivative:

$$d(M\Omega) = dM \wedge \Omega + M d\Omega$$

Therefore we have

$$d\eta_0 = dM_1 \wedge \omega_1 + dM_2 \wedge \omega_2.$$

Also of easy verification we have

$$\eta_0 \wedge \eta_0 = [M_1, M_2] \omega_1 \wedge \omega_2$$

where $[\cdot, \cdot]$ denotes the matrix Lie bracket. Thus we obtain

$$dM_1 \wedge \omega_1 + dM_2 \wedge \omega_2 = [M_1, M_2] \omega_1 \wedge \omega_2.$$

Taking the exterior product with ω_2 in the above equation we obtain

$$dM_1 \wedge \omega_1 \wedge \omega_2 = 0$$

Hence, M_1 is a meromorphic first integral for the foliation defined by the system $\{\omega_1, \omega_2\}$ in $\tilde{U} := U' \setminus \{x_1 x_2 x_3 = 0\}$. This foliation is exactly the restriction of \mathcal{F} to this open set. Since \mathcal{F} is defined by the vector field X in \tilde{U} and this vector field is linear without resonance, it follows from Lemma 5.1 that M_1 is constant in \tilde{U} . Similarly we can conclude that M_2 is constant. This implies the extension result and the other items in Claim 6.1. \square

The proof of Lemma 6.1 is complete. \square

Proof of Theorem 6.1. The proof follows the same argumentation as the proof of Lemma 3.2 in Chapter I in [13]. Indeed, Lemma 6.1 implies that η extends meromorphically to $\Lambda \cup \text{sep}(\Lambda)$. By construction this extension is adapted to Ω along Λ . \square

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